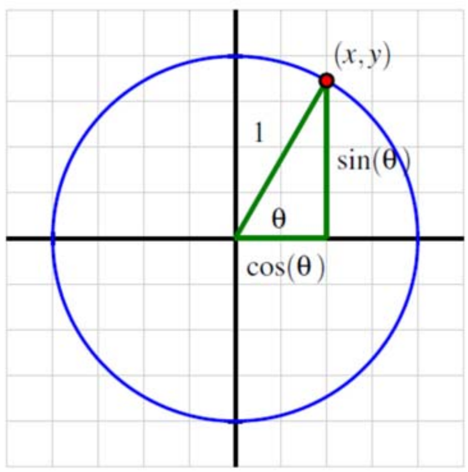
2.24 Tan(z) and the Reciprocals

Derivatives of Other Trigonometric Functions

One of the powerful themes in trigonometry is that the entire subject emanates from a very simple idea: locating a point on the unit circle.



Because each angle θ corresponds to one and only one point (x, y) on the unit circle, the x- and y-coordinates of this point are each functions of θ. Indeed, this is the very definition of $\cos(\theta)$ and $\sin(\theta)$: $\cos(\theta)$ is the x-coordinate of the point on the unit circle corresponding to the angle θ, and $\sin(\theta)$ is the y-coordinate. From this simple definition, all of trigonometry is founded. For instance, the fundamental trigonometric identity,

$$\sin^2(\theta) + \cos^2(\theta) = 1,$$

is a restatement of the Pythagorean Theorem, applied to the right triangle shown on the unit circle above.

Recall there are four other trigonometric functions, each defined in terms of the sine and/or cosine functions. These six trigonometric functions together offer you a wide range of flexibility in problems involving right triangles. The tangent function is defined by $\tan\theta = \frac{\sin\theta}{\cos\theta}$, while the cotangent function is its reciprocal: $\cot\theta = \frac{\cos\theta}{\sin\theta}$

The secant function is the reciprocal of the cosine function, $\sec\theta = \frac{1}{\cos\theta}$, and the cosecant function is the reciprocal of the sine function, $\csc\theta = \frac{1}{\sin\theta}$

Because you know the derivatives of the sine and cosine function, and the other four trigonometric functions are defined in terms of these familiar functions, you can now develop shortcut differentiation rules for the tangent, cotangent, secant, and cosecant functions. In this section's preview activity, you work through the steps to find the derivative of $y = \tan(x)$.

Investigation 1: Consider the function $f(x) = \tan(x)$, and remember that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

a) What is the domain of f?

b) Use the quotient rule to show that one expression for $f'(x)$ is

$$f'(x) = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{cos^2(x)}$$

c) What is the Fundamental Trigonometric Identity? How can this identity be used to find a simpler form for $f'(x)$?

d) Recall that $\csc(x) = \frac{1}{\sin(x)}$. How can you express $f'(x)$ in terms of the secant function?

e) For what values of x is $f'(x)$ defined? How does this set compare to the domain of f?

II. Derivatives of the cotangent, secant, and cosecant functions

In Investigation 1, you found that the derivative of the tangent function can be expressed in several ways, but most simply in terms of the secant function. Next, you will meet the derivative of the cotangent function.

Let g(x) = cot(x). To find g'(x), you observe that $g(x) = \frac{\cos(x)}{\sin(x)}$ and apply the quotient rule. Hence

$$g'(x) = \frac{\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)}$$

$$= \frac{\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$

By the Fundamental Trigonometric Identity, you get

$$= \frac{-1}{\sin^2(x)}$$

It follows that you can most simply express g, by the rule

$$g'(x) = -\csc^2(x)$$

Note that neither g nor g' is defined when $\sin(x) = 0$, which occurs at every integer multiple of π. Hence you have the following rule.

Tangent and Cotangent Functions: For all real number x, such that $x \neq k\pi$, where $k = 0, \pm1, \pm2, \pm3 \ldots$,

- $\frac{d}{dx}[\tan(x)] = \sec^2(x)$ and
- $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$,

In the next two activities, you will develop the rules for differentiating the secant and cosecant functions.

Investigation 2: Let $h(x) = \sec x$ and recall that $\sec(x) = \frac{1}{\cos(x)}$.

a) What is the domain of h?

b) Use the quotient rule to develop a formula for $h'(x)$ that is expressed completely in terms of $\sin(x)$ and $\cos(x)$.

c) How can you use other relationships among trigonometric functions to write $h'(x)$ only in terms of $\tan(x)$ and $\sec(x)$?

d) What is the domain of h'? How does this compare to the domain of h?

Investigation 3: Let $p(x) = \csc(x)$ and recall that $\csc(x) = \frac{1}{\sin(x)}$.

a) What is the domain of p?

b) Use the quotient rule to develop a formula for $p'(x)$ that is expressed completely in terms of $\sin(x)$ and $\cos(x)$.

c) How can you use other relationships among trigonometric functions to write $p'(x)$ only in terms of $\cot(x)$ and $\csc(x)$?

d) What is the domain of p'? How does this compare to the domain of p?

The quotient rule has thus enabled you to determine the derivatives of the tangent, cotangent, secant, and cosecant functions, expanding your overall library of basic functions you can differentiate. Moreover, you observe that just as the derivative of any polynomial function is a polynomial, and the derivative of any exponential function is another exponential function, so it is that the derivative of any basic trigonometric function is another function that consists of basic trigonometric functions. This makes sense because all trigonometric functions are periodic, and hence their derivatives will be periodic, too.

As has been and will continue to be the case throughout your work in Big Idea 2 - Derivatives, the derivative retains all of its fundamental meaning as an instantaneous rate of change and as the slope of the tangent line to the function under consideration. Your present work primarily expands the list of functions for which you can quickly determine a formula for the derivative. Moreover, with the addition of tan (x), cot (x), sec (x), and csc (x) to your library of basic functions, there are many more functions you can differentiate through the sum, constant multiple, product, and quotient rules.

Investigation 4: Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

a) Let $f(x) = 5\sec(x) - 2\csc(x)$. Find the slope of the tangent line to f at the point where $x = \frac{\pi}{3}$.

b) Let $p(z) = z^2 \sec(z) - z\cot(z)$. Find the instantaneous rate of change of p at the point where $z = \frac{\pi}{4}$

c) Let $h(t) = \frac{\tan(t)}{t^2+1}$. Find $h'(t)$.

d) Let $g(r) = \frac{r\sec(r)}{5^r}$. Find $g'(r)$.

e) When a mass hangs from a spring and is set in motion, the object's position oscillates in a way that the size of the oscillations decrease. This is usually called a ***damped oscillation***. Suppose that for one such object, the displacement from equilibrium (where the object sits at rest) is modeled by the function

$$s(t) = \frac{15\sin(t)}{e^t}.$$

Assume that s is measured in inches and t in seconds. Sketch a graph of this function for $t \geq 0$ to see how it represents the situation described. Then compute $\frac{ds}{dt}$, state the units on this function, and explain what it tells you about the object's motion. Finally, compute and interpret $s'(2)$.

III. Exercises

1. An object moving vertically has its height at time t (measured in feet, with time in seconds) given by the function $h(t) = 3 + 2\cos(t)$.

a) What is the object's instantaneous velocity when $t = 2$?

b) What is the object's acceleration at the instant $t = 2$?

c) Describe in everyday language the behavior of the object at the instant $t = 2$.

2. Let $f(x) = \sin(x)\cot(x)$.

a) Use the product rule to find $f'(x)$.

b) True or false: for all real numbers x, $f(x) = \cos(x)$.

c) Explain why the function that you found in (a) is almost the opposite of the sine function, but not quite. (Hint: convert all of the trigonometric functions in (a) to sines and cosines, and work to simplify. Think carefully about the domain of f and the domain of f'.)

3. Let $p(z)$ be given by the rule

$$p(z) = \frac{z\tan(z)}{z^2\sec(z) + 1}$$

a) Determine $p'(z)$.

b) Find an equation for the tangent line to p at the point where $z = 0$.

c) At $z = 0$, is p increasing, decreasing, or neither? Why?

IV. Practice – Khan Academy

1. Complete the following online practice exercises in the Differentiating Common Functions unit of Khan Academy's AP Calculus AB course:
   a. https://www.khanacademy.org/math/ap-calculus-ab/differentiating-common-functions-ab/trigonometric-functions-differentiation-ab/e/differentiate-basic-trigonometric-functions
   b. https://www.khanacademy.org/math/ap-calculus-ab/differentiating-common-functions-ab/trigonometric-functions-differentiation-ab/e/differentiate-trigonometric-functions
2. Optional: none